EDITED BY MELVIN HENRIKSEN AND STAN WAGON

A Euclidean Model for Euclidean Geometry

ADOLF MADER
Department of Mathematics, University of Hawaii, Honolulu, HI 96822

I. Introduction. In courses on non-Euclidean geometry models of hyperbolic geometry play an important role. The standard models are Klein's disk model and the disk model and half-plane model named after Poincaré [1], [4], [6], [7], [8]. In each case the model is embedded in the Euclidean plane and the concepts of hyperbolic geometry are interpreted in Euclidean terms. One wonders whether there is a similar model of Euclidean geometry within Euclidean geometry, and in particular a model that is bounded as a Euclidean object. Then for once one could see the whole Euclidean world that tends to run off the blackboard all too fast. A little reflection yields the correct answer: Of course! After all, all that is needed is a bijection of the standard Euclidean space with a bounded part of itself, and by means of this bijection the structure of Euclidean space can be transferred to the model. Doing this randomly will likely result in a model which is quite intractable without going back to the original. We will below propose a model that is reasonably simple in itself and has a transparent relationship with standard Euclidean space. It demonstrates the difference between the logical content of an axiom system and its interpretations; in particular it destroys the faith in a preordained single model of Euclidean space. It can serve as an introduction to the idea of models of a geometry, and thus make the models of unfamiliar geometries more palatable. It further constitutes a source of questions and exercises, and looks like a pedagogically convenient way to introduce the projective extension of Euclidean space.

While there was never a doubt that this model must have appeared someplace before, I only discovered it in the textbook by David Gans [3] when I revised my article. Gans uses the model extensively for motivating and illustrating projective geometry. It is introduced on page 212 after the study of transformations of various kinds. However, the model can be used profitably from the start in any geometry course and deserves more publicity. I would like to thank the referee for a thorough review and many helpful suggestions.

II. The Model $\mathbb{F}$. Let $\mathbb{F}$ be the interior of a circle ω of radius r in the standard Euclidean plane $\mathbb{E}$. The straight lines in $\mathbb{F}$ are the half-ellipses of $\mathbb{E}$ whose major axes are diameters of ω and the diameters themselves. It is convenient to include among the "half-ellipses" the diameters of ω, which then coincide with their "major axes" and have degenerate (one-point) "minor axes." At this stage we may observe:

(2.1) The families of parallel lines in $\mathbb{F}$ are the families of half-ellipses with common major axes.

The angle measure between two straight lines of $\mathbb{F}$ is the $\mathbb{E}$-measure of the angle between their major axes. The $\mathbb{F}$-distance $\bar{d}$ between the center O of ω and a point of $\mathbb{F}$ with $\mathbb{E}$-distance d from O is

(2.2) $$\bar{d} = rd\Big/\sqrt{r^2 - d^2}\,.$$

The distance between any two points on a diameter is computed from their distances from O, and the distance between two arbitrary points can be computed by transferring it to a diameter by means of a parallelogram.

2.3 THEOREM. $\mathbb{F}$ *is a model of the Euclidean plane.*

The validity of 2.3 is established by means of the following map.

(2.4) The bijection $\sigma\colon \mathbb{E} \to \mathbb{F}$. The plane $\mathbb{E}$ is contained in 3-dimensional Euclidean space and we choose Cartesian coordinates in the latter such that $\mathbb{E}$ is the x–y plane, ω is the circle $x^2 + y^2 = r^2$, and $\mathbb{F}$ is the interior of ω. Let S be the hemisphere $x^2 + y^2 + (z - r)^2 = r^2$, $z < r$, with center C. For the initiated, σ is simply the central or gnomonic projection from $\mathbb{E}$ onto S composed with the orthogonal projection of S onto ω [1], [2]. Explicitly, given $P \in \mathbb{E}$, let P_1 be the point of intersection of the line PC with S and let $\bar{P} = \sigma(P)$ be the intersection with $\mathbb{E}$ of the parallel to the z-axis through P_1.

It is clear that this map $\sigma\colon \mathbb{E} \to \mathbb{F}$ is a bijection. The model $\mathbb{F}$ is established by using σ to transfer to it the structure of $\mathbb{E}$. The image $\sigma(a)$ of a straight line a of $\mathbb{E}$ is obtained by intersecting the plane through a and C with S which results in a great semicircle a_1. Projecting a_1 into $\mathbb{F}$ yields an ellipse whose principal axis is the projection of the diameter of a_1 in $z = r$ (Fig. 1). It is now clear that angles in $\mathbb{F}$ are measured as described above. In terms of coordinates σ is given as follows.

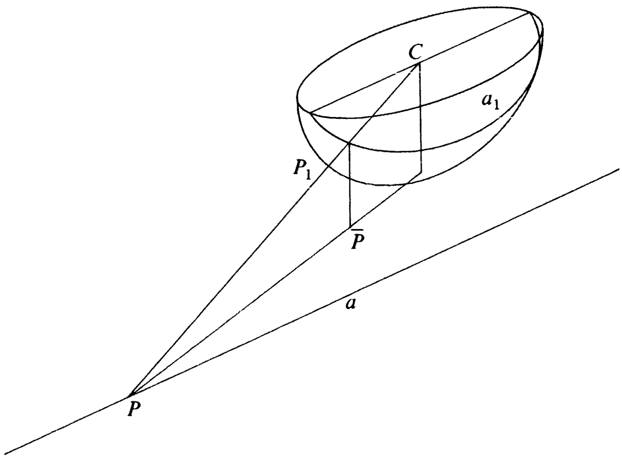

FIG. 1

(2.5) Let P be a point of $\mathbb{E}$ and let $\bar{P} = \sigma(P)$. If (ρ, φ) are the polar coordinates of P and $(\bar{\rho}, \bar{\varphi})$ the polar coordinates of $\bar{P}$ then

$$\bar{\rho} = r\rho\Big/\sqrt{r^2 + \rho^2}\,, \qquad \bar{\varphi} = \varphi,$$

$$\rho = r\bar{\rho}\Big/\sqrt{r^2 - \bar{\rho}^2}\,, \qquad \varphi = \bar{\varphi}.$$

If (x, y) are the Cartesian coordinates of P and $(\bar{x}, \bar{y})$ the Cartesian coordinates of $\bar{P}$ then

$$\bar{x} = rx\Big/\sqrt{r^2 + x^2 + y^2}, \qquad \bar{y} = ry\Big/\sqrt{r^2 + x^2 + y^2},$$
$$x = r\bar{x}\Big/\sqrt{r^2 - \bar{x}^2 - \bar{y}^2}, \qquad y = r\bar{y}\Big/\sqrt{r^2 - \bar{x}^2 - \bar{y}^2}.$$

Proof. In Fig. 2, $OP = \rho$, $O\bar{P} = \bar{\rho}$ and $\bar{P}P_1 = r - \sqrt{r^2 - \bar{\rho}^2}$ from the equation of the sphere. Similar triangles yield

$$\frac{\rho}{r} = \frac{\rho - \bar{\rho}}{r - \sqrt{r^2 - \bar{\rho}^2}},$$

whence $\rho = r\bar{\rho}\Big/\sqrt{r^2 - \bar{\rho}^2}$. The rest follows easily.

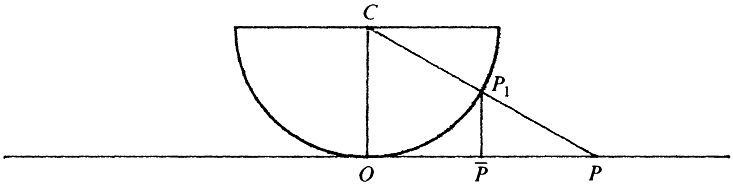

FIG. 2

The formulae (2.5) together with the fact that $\mathbb{F}$ is a Euclidean geometry (being a carbon copy of $\mathbb{E}$) show that the distance measures in $\mathbb{F}$ are obtained as claimed, and 2.3 is now proven.

Another model of Euclidean geometry has appeared partway through the above proof. The central projection maps the Euclidean plane $\mathbb{E}$ bijectively onto the hemisphere

$$\mathbb{H}: x^2 + y^2 + (z - r)^2 = r^2, \qquad z < r,$$

transforming straight lines into great semicircles. The model $\mathbb{F}$ may be viewed as a perspective picture of the hemispherical model $\mathbb{H}$.

Translating properties of $\mathbb{F}$ into the language of $\mathbb{E}$ yields facts and construction problems such as the following.

(2.6) Given a circle ω and two points in its interior, there exists a unique ellipse passing through the given points and having as major axis a diameter of ω. Find a straightedge and compass construction of the major and minor axis of this ellipse.

(2.7) Given a circle ω, a diameter of ω, and a point in the interior of ω, there exists a unique ellipse having the given diameter as major axis and passing through the given point. Find a straightedge and compass construction of the minor axis of this ellipse.

(2.8) Let A be an ellipse whose major and minor axes are given. Let a be a line passing through the center of A. Find a straightedge and compass construction of the points of intersection of a and A.

(2.9) Let A_1 and A_2 be concentric ellipses with known major and minor axes. If the major axes of A_1 and A_2 have equal lengths, find a straightedge and compass construction of the points of intersection of A_1 and A_2.

These construction problems can be solved by mapping the $\mathbb{F}$-lines back to $\mathbb{E}$ and thereby turning the ellipses into straight lines. Figure 3 indicates how P can be constructed from $\bar{P} = \sigma(P)$ and conversely.

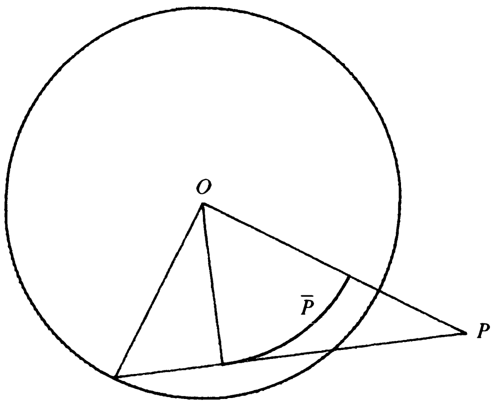

FIG. 3

III. The Global Picture. It is now possible to illustrate Euclidean curves in their entirety in $\mathbb{F}$. FIGURES 4–7 illustrate the kinds of graphs one gets in $\mathbb{F}$. In each figure the radius of $\mathbb{F}$ is 12 units.

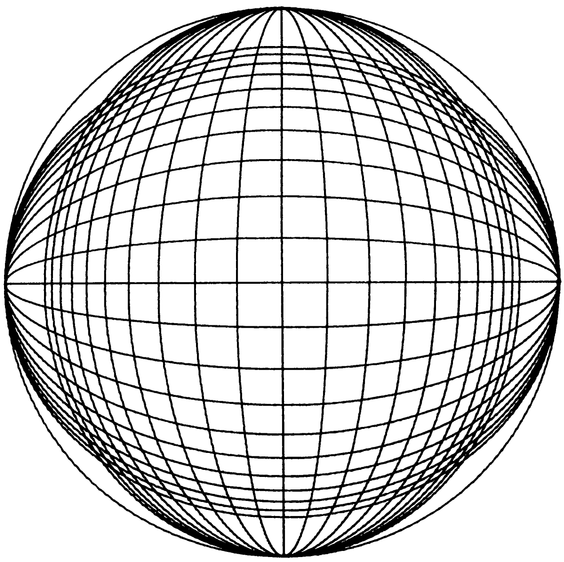

FIG. 4. Coordinate grid $x = 2n$, $y = 2n$, $-10 \leqslant n \leqslant +10$.

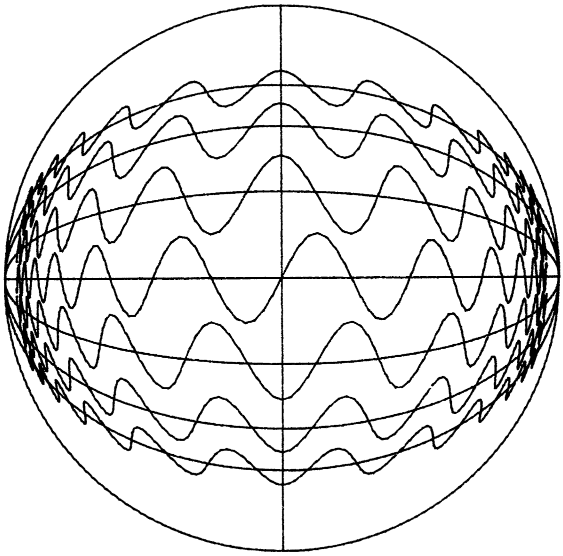

FIG. 5. Sine and cosines $y = 2\sin x$, $y = \pm 2(2n + \cos x)$ $y = \pm 4n, 1 \leqslant n \leqslant 3$

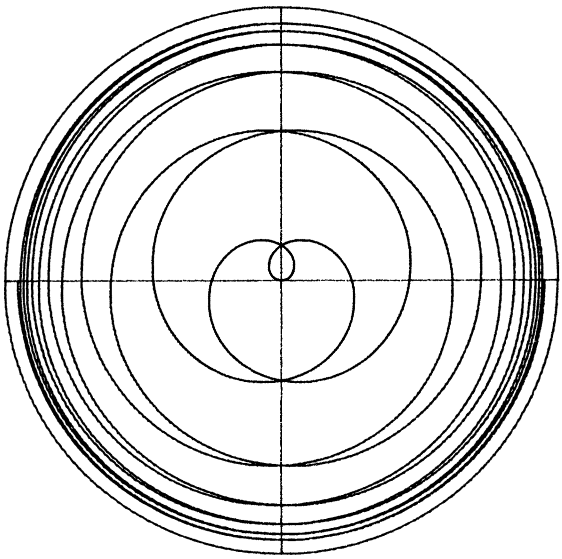

FIG. 6. The Archimedean spiral $\rho = \varphi$.

As in the case of the models of hyperbolic geometry the model $\mathbb{F}$ is deceptive in suggesting a distinguished center of the Euclidean plane. This center is distinguished only from the point of view of the ambient space but indistinguishable from other points from within the model.

The coordinate grid (FIG. 4) is readily recognized as the perspective picture of the corresponding family of great semicircles of the hemispherical model $\mathbb{H}$. It is a bit more challenging to imagine the preimages on the hemisphere of the other figures.

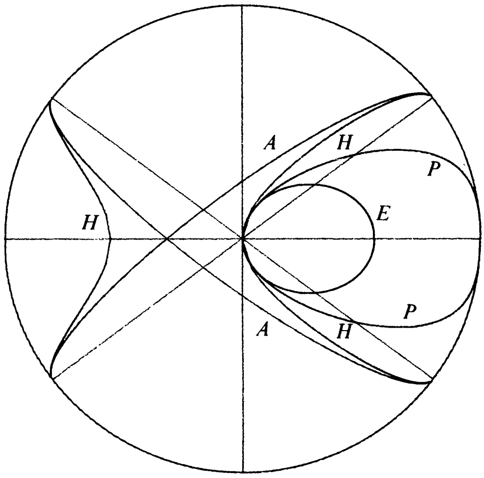

FIG. 7. Conic sections.

E: Ellipse $\frac{(x-4)^2}{16} + \frac{y^2}{9} = 1$, P: {Parabola $y^2 = 4x$
H: Hyperbola $\frac{(x+4)^2}{16} - \frac{y^2}{9} = 1$ with asymptotes A: $y = \pm\frac{3}{4}(x+4)$

IV. Projective Geometry. The ideal points that must be added to $\mathbb{F}$ in order to get the real projective plane $\mathbb{P}$ are there for all to see. They are the points of ω, and ω is the line at infinity. A family of parallels meets at the endpoints of their common principal axes, and at least at this stage we see that the two endpoints must be identified in order to preserve the incidence axioms. Having these ideal points available, Fig. 7 suggests that the distinguishing features of ellipses, parabolas, and hyperbolas are merely that they have none, one, or two points in common with the ideal line ω. Asymptotic curves meet on ω but the converse need not be true as we can see from the parabola.

The projective closure of the hemispherical model $\mathbb{H}$ is the closed hemisphere

$$x^2 + y^2 + (z - r)^2 = r^2, \qquad z \leqslant r,$$

with antipodal points on the equator identified. In this setting an angle and distance measure can be defined conveniently [**1**], [**6**], [**7**] yielding a standard model of elliptic geometry. More models of various geometries can be found using the full sphere and utilizing stereographic projection [**1**], [**5**].

V. Extension to Higher Dimensions. There is no difficulty at all to extending the isomorphism (2.5) to arbitrary dimension n. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote direction angles, $\rho, \bar{\rho}$ the distances from the origin, and $x_i, \bar{x}_i$ the Cartesian coordinates of corresponding points P and $\bar{P}$ respectively then the transformation equations are:

$$\bar{\rho} = \frac{r\rho}{\sqrt{r^2 - \rho^2}}, \qquad \bar{\alpha}_i = \alpha_i; \qquad \rho = \frac{r\bar{\rho}}{\sqrt{r^2 - \bar{\rho}^2}}, \qquad \alpha_i = \bar{\alpha}_i. \tag{5.1}$$

$$\bar{x}_i = \frac{rx_i}{\sqrt{r^2 + \Sigma x_i^2}}; \qquad x_i = \frac{r\bar{x}_i}{\sqrt{r^2 - \Sigma \bar{x}_i^2}}. \tag{5.2}$$

The representative hyperplane $x_1 = A$ is transformed into the half-ellipsoid $(r^2 + A^2)x_1^2 + A^2x_2^2 + \cdots + A^2x_n^2 = A^2r^2$, $x_1 \cdot A \geqslant 0$.

VI. Philosophical Implications. We all know the murals in churches in which the deity is surrounded by angels and looking down onto earth from his seat on the clouds. It seems like a rather naive image now when every child knows that the immensity of space only begins beyond the clouds, and we expect to encounter strange creatures of somewhat human forms on space ventures rather than God and angels. However, slight corrections in the traditional pictures could create depictions which are mathematically perfectly consistent: simply enter the impenetrable shell of ideal points between the world and the heavens. Even in a Euclidean universe there is plenty of space for one or many worlds like ours and still more for heaven and hell—it all depends on how you measure.

REFERENCES

1. H. S. M. Coxeter, Introduction to Geometry, John Wiley & Sons, New York, 5th Printing, 1966.
2. H. S. M. Coxeter and S. L. Greitzer, Geometry Revisited, New Mathematical Library No. 19, Mathematical Association of America, Washington, 1967.
3. D. Gans, Transformations and Geometries, Appleton-Century-Crofts, New York, 1969.
4. M. J. Greenberg, Euclidean and Non-Euclidean Geometries, 2nd ed., W. H. Freeman and Company, San Francisco, 1980.
5. N. W. Johnson, Absolute polarities and central inversions, in The Geometric Vein: The Coxeter Festschrift, C. Davis, B. Grünbaum, F. A. Sherk, eds., Springer Verlag, New York, 1981, 443–464.
6. H. Meschkowski, Non-Euclidean Geometry, Academic Press, New York and London, 1964.
7. E. Moise, Elementary Geometry from an Advanced Standpoint, Addison-Wesley, Reading, Mass. 1963.
8. C. R. Wylie, Jr., Foundations of Geometry, McGraw-Hill, New York, 1964.

Orthogonal Bases of $\mathbb{R}^3$ with Integer Coordinates and Integer Lengths

ANTHONY OSBORNE AND HANS LIEBECK
Department of Mathematics, University of Keele, Staffordshire ST5 5BG, England

An interesting problem that arises when studying orthogonal bases of $\mathbb{R}^3$ is to find such bases of vectors with integer coordinates and integer lengths. The basis $\{(2,2,-1), (2,-1,2), (-1,2,2)\}$ and its relatives feature prominently in textbooks, but there is a shortage of other examples. In this note we give a complete solution of the problem.

It clearly suffices to consider the *primitive* bases $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ of $\mathbb{R}^3$ where each vector $\mathbf{u},\mathbf{v},\mathbf{w}$ has integer length and the three coordinates of $\mathbf{u},\mathbf{v},\mathbf{w}$ are, in each case, relatively prime integers. The general solution of our problem is then obtained from the bases $\{a\mathbf{u}, b\mathbf{v}, c\mathbf{w}\}$, where a, b and c are arbitrary nonzero integers.

A simple construction. The following simple construction provides a rich source of orthogonal bases of the type we want. Consider nonzero vectors in $\mathbb{R}^3$ with